TranSust.Scan WP 1.2.3: Sensitivity Analysis

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The Task

Evaluating the sensitivity of model simulations task 3 will involve:

- Deterministic approaches for sensitivity analysis based on varying the key parameters within a defined range.
- Stochastic approaches for sensitivity analysis based on Monte Carlo and Gaussian quadrature type methods.

from Workpackage 1.2, Task 3

Motivation

"A fragile inference is not worth taking seriously. All scientific disciplines routinely subject their inferences to studies of fragility. Why should economics be different? ... What we need are organized sensitivity analyses. [...] We need to be shown that minor changes in the list of variables do not alter fundamentally the conclusions."

Edward E. Leamer (AER, 1985)

Motivation

Definition

Sensitivity analysis is the study of how the variation in the output of a model (numerical or otherwise) can be apportioned, qualitatively or quantitatively, to different sources of variation.

from: Wikipedia

Mathematical Preliminaries

An equilibrium of a CGE model takes the mathematical form of a solution to a system of (non-linear) equations

$$G(x^*, a) = 0,$$

where $x^* \in \mathbf{R^n}$ is a vector of (equilibrium) state variables of the economy (such as capital or wage) and $a \in \mathbf{R^d}$ a vector of parameters of the economy (such as demand elasticity or time preference) and $G : \mathbf{R^n} \to \mathbf{R^n}$ is continously differentiable.

Mathematical Preliminaries

Theorem (Implicit function theorem)

If det $|\nabla_{x^*} G(x^*, a)| \neq 0$, then there exists an open neighbourhood $\mathcal{U}(a) \subset \mathbf{R^d}$ of a and a continously differentiable function $h : \mathcal{U}(a) \rightarrow \mathbf{R^n}$ that maps any vector of parameters on the corresponding equilibrium vector.

The implicit function theorem applies whenever the CGE model has a unique equilibrium.

Mathematical Preliminaries

BEWARE OF MULTIPLE EQUILIBRIA!

In this case, $h : \mathcal{U}(a) \rightarrow \mathcal{P}(\mathbf{R^n})$ is a correspondence, mapping the vector of parameters a into a set of solutions $\{x_1^*, x_2^*, ..., x_m^*\}$. Ignoring multiplicity can seriously blur a sensitivity analysis.

Deterministic Sensitivity Analysis

The **deterministic approach** to sensitivity analysis states is that there exists one true vector of economic parameters $a^* \in \mathbf{R^d}$, but that -instead of a^*- we only know its neighbourhood $\mathcal{A}$. Usually, we choose one vector of parameters $\hat{a} \in \mathcal{A}$ (*benchmark scenario*). We want to know whether equilibria vary considerably across $h(\mathcal{A})$ in comparison to the equilibrium $h(\hat{a})$.

Deterministic Sensitivity Analysis

Mathematically speaking, a sensitivity analysis asks whether relation of the volume of the image of $\mathcal{A}$ under h and the size of $h(\hat{a})$, weighted with a scaling factor w_k in each dimension,

$$\frac{vol(im(w * h))}{\|w * h(\hat{a})\|} = \frac{\int_{\mathcal{A}} \sqrt{\det(\partial_{ij}(w * h(a)))} da_1 ... da_d}{(w * h(\hat{a}), w * h(\hat{a}))},$$

is sufficiently small. The vector w specifies the relative weight we want to attach to the different economic variables in equilibrium.

Stochastic Sensitivity Analysis

The **stochastic approach** to sensitivity analysis treats the vector of parameters as a stochastic variable a with a given distribution $G(a)$ of $a \in \mathcal{A}$. Consequently, h becomes a mapping onto a stochastical variable $x^* = h(a)$ of equilibria.

Stochastic Sensitivity Analysis

We then calculate the mean and the variation of the equilibrium vector x^*:

$$m = E[h(a)] = \int_{\mathcal{A}} h(a)dG,$$

$$v = Var[h(a)] = E[(h(a) - m)^2] = \int_{\mathcal{A}} (h(a) - m)^2 dG.$$

The stochastic sensitivity analysis asks for the size of

$$\sum w_k \frac{v_k}{m_k}.$$

The Peacemeal Approach

In a **peacemeal approach** to sensitivity analysis, we calculate

$$\Delta = \max_{a_i, a_j \in \{a_1, \ldots, a_M\}} |h(a_i) - h(a_j)|$$

for a set of representative parameters $a_i \in \mathcal{A}$. The relation of Δ to the weighted benchmark equilibrium $h(\hat{a})$ is used to assess the sensitivity of the model at equilibrium $h(\hat{a})$.

In the following, w.l.o.g. let $n = 1$.

The Peacemeal Approach

A peacemeal approach can give a good idea of the sensitivity of the model if the set of parameters $a_i \in \mathcal{A}$ is sufficiently representative. If the set contains $\underline{a} = \mathbf{argmin}_{\mathbf{a} \in \mathcal{A}} \mathbf{h}(\mathbf{a})$ and $\overline{a} = \mathbf{argmax}_{\mathbf{a} \in \mathcal{A}} \mathbf{h}(\mathbf{a})$, then the following inequality holds:

$$\frac{vol(im(h))}{|h(\hat{a})|} \leq vol(\mathcal{A}) \frac{h(\overline{a}) - h(\underline{a})}{|h(\hat{a})|} = vol(\mathcal{A}) \frac{\Delta}{|h(\hat{a})|}.$$

The Peacemeal Approach

As an example, let both $n = 1$ and $d = 1$. Then

$$h : [\underline{a}, \overline{a}] \rightarrow [\underline{x}, \overline{x}]$$

for some scalar parameters $\underline{a}$, $\overline{a}$, $\underline{x}$ and $\overline{x}$. If h is monotonously increasing, then

$$\frac{vol(im(h))}{h(\hat{a})} \leq (\overline{a} - \underline{a}) \frac{h(\overline{a}) - h(\underline{a})}{h(\hat{a})}.$$

The Monte-Carlo Approach

In a **Monte-Carlo** approach, we approximate both mean and variance of equilibrium x^*. We draw a set of realisations $\{a_1, ..., a_M\}$ from the distribution $G(\mathcal{A})$ and calculate

$$m = E[h(a)] \approx \frac{1}{M} \sum_{i=1}^{M} h(a_i) = \tilde{m},$$

$$v = Var[h(a)] \approx \frac{1}{M} \sum_{i=1}^{M} (h(a_i) - \tilde{m})^2 = \tilde{v}.$$

The Monte-Carlo Approach

Advantage of Monte-Carlo approach: Beyond mean and variance of the stochastic variable x^*, we can approximate its distribution $h \circ G$.

Disadvantage of Monte-Carlo approach: To assure convergence, M has to be high and so the approximation is numerically costly.

The Gauss-Quadrature Approach

Gauss quadrature is a numerical method to approximate integrals. We would like to approximate mean and variance, using a small number L of evaluations of $h(.)$.

Essentially, the Gauss quadrature gives us nodes x_i and weights ω_i to approximate the (one dimensional) integral

$$\int_a^b f(x)\omega(x)dx \approx \sum_{i=1}^{L} \omega_i f(x_i).$$

The Gauss-Quadrature Approach

In our case, we look for nodes a_i and weights g_i to approximate mean and variance of equilibria (dimension $n = 1$):

$$m = \int_{\mathcal{A}} h(a) dG \approx \sum_{i=1}^{L} g_i h(a_i) = \tilde{m},$$

$$v = \int_{\mathcal{A}} (h(a) - m)^2 dG \approx \sum_{i=1}^{L} g_i (h(a_i) - \tilde{m})^2 = \tilde{v}.$$

The Gauss-Quadrature Approach

Let the distribution $G(a)$ be represented by a weight function $g(a)$. Then the expression $(f_1, f_2)_g = \int_{\mathcal{A}} f_1(a) f_2(a) g(a) da$ defines a scalar product $(.,.)_g$. The following Lemma holds:

Lemma (Gram-Schmidt, Weierstrass)

For any scalar product $(.,.)$ on the space of continuous functions $\mathcal{C}([\underline{a}, \overline{a}])$, there is a complete system of orthogonal polynomials $\{p_0, p_1, ...|(p_i, p_j) = 0, i \neq j\}$.

The Gauss-Quadrature Approach

For a given scalar product, orthogonal polynomials can be constructed from monomials 1, x, x^2,... by the Gram-Schmidt procedure

$$p_0 \equiv 1 \qquad p_i(x) = x^i - \sum_{j=1}^{i-1} \frac{(p_j, x^i)}{(p_j, p_j)} p_j .$$

Furthermore, we need one property of orthogonal polynomials.

Lemma

The zeros $\{a_1, a_2, ..., a_I\}$ of $p_I(a)$ are real and distinct.

Practical Sensitivity Analysis

Examples of families of orthogonal polynomials:

Name	$g(x)$	$[a,b]$	Definition
Legendre	1	$[-1,1]$	$P_k(x) = \frac{(-1)^k}{2^k k!}\frac{d^k}{dx^k}[(1-x^2)^k]$
Tschebyscheff	$(1-x^2)^{-\frac{1}{2}}$	$[-1,1]$	$T_k(x) = \cos(k\cos^{-1}(x))$
Laguerre	$\exp(-x)$	$[0,\infty)$	$L_k(x) = \frac{\exp(x)}{k!}\frac{d^k}{dx^k}(x^k\exp(-x))$
Hermite	$\exp(-x^2)$	$(-\infty,\infty)$	$H_k(x) = (-1)^k\exp(x^2)\frac{d^k}{dx^k}(\exp(-x^2))$

The Gauss-Quadrature Approach

Theorem (Stoer)

Let $\{a_1, a_2, ..., a_l\}$ *be the zeros of* $p_l(a)$ *and* $g_1, ..., g_l$ *be the solution of the system of linear equations*

$$\sum_{i=1}^{n} g_i p_k(a_i) = \begin{cases} (p_0, p_0) & : \quad k = 0 \\ 0 & : \quad k = 1, 2, ..., l-1 \end{cases}$$

Then $g_i > 0$ *for* $i = 1, 2, ..., l$ *and*

$$\int_{\underline{a}}^{\overline{a}} p(a) g(a) da = \sum_{i=1}^{l} g_i p(a_i)$$

for all $p \in \Pi_{2l-1} = \lll p_0, ..., p_{2l-1} \ggg$.

The Gauss-Quadrature Approach

In words: For a given weight function $g(a)$ (i.e. probability distribution G), we can calculate nodes $a_1, ..., a_l$ and weights $g_1, ..., g_l$, so that polynomials up to degree 2l-1 are integrated exactly.

For our original purpose, we have to calculate the zeros of orthogonal polynomials and weights corresponding to the probability distribution G with weight function $g(a)$.

However, we have to do so only once for a given G.

The Gauss-Quadrature Approach

In higher dimensions d, integrals can be approximated by product rules, combining one-dimensional nodes and weights:

$$\int_{\underline{a}^1}^{\overline{a}^1} \dots \int_{\underline{a}^d}^{\overline{a}^d} f(a^1, \dots, a^d) g^1(a^1) \dots g^d(a^d) da^d \dots da^1$$
$$\approx \sum_{i_1=1}^{n} \dots \sum_{i_d=1}^{n} g^1_{i_1} \dots g^d_{i_d} f(a^1_{i_1}, \dots, a^d_{i_d})$$

A Simple Example

We implement a simple example in Markusen's spirit (2002):

CGE model with

- two goods X and Y
- two input factors, capital K and labor L
- factor labor taxes on both goods, $\mathtt{TAX}_{\mathtt{L}}\mathtt{X}$ and $\mathtt{TAX}_{\mathtt{L}}\mathtt{Y}$
- Hicksian welfare measure W

A Simple Example

We have three economic parameters with respect to which we conduct a sensitivity analysis:

- $\texttt{esubx}$ elas. of subst. between inputs to X production
- $\texttt{esuby}$ elas. of subst. between inputs to Y production
- $\texttt{esubw}$ elas. of subst. between inputs to final demand

We choose $\texttt{esubx} = \texttt{esuby} = \texttt{esubw} = \texttt{0.5}$ and vary each of the parameters uniformly between 0.25 and 0.75.

A Simple Example

We have calculated mean and variance of all economic variables by

- Monte-Carlo analysis
- Gauss-Quadrature with Legendre polynomials

A Simple Example

GQ algorithm:

We have calculated the zeros of Legendre polynomials `legendrenodes` and the weights `legendreweights` in a MATLAB routine and saved them in `gdx` files.

In a GAMS program, the variables `legendrenodes` and `legendreweights` are loaded and transformed into the variables `grid` and `weights`.

A Simple Example

`GAMS` program

1. Implement CGE model with given `esubx`, `esuby`, `esubw`
2. Transform `legendrenodes` linearly from interval $[-1, 1]$ to `grid` on interval $[0.25, 0.75]$
3. Calculate `results` by a loop of the model, varying elasticity/elasticities across `grid`
4. Calculate mean and variance of economic variables by summing up `results`, weighted by `weights`

A Simple Example

Sensitivity w.r.t. `esubx`: Results of MC and GQ

Number of evaluations: $M_{MC} = 100$, $l_{GQ} = 10$

Name	MC mean	GQ mean	MC var	GQ var
X	-8.990	-9.436	0.027	0.021
Y	7.582	7.547	0.004	0.004
W	-1.396	-1.679	0.014	0.043

References

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